$$H^1(S^{\mathrm{Assy}(3)}(G_K), \triangle_\Theta \otimes \mathbb{Z}/\ell)$$

('Recall) $\left(s^{\mathrm{Assy}(3)}(G_K) \ni s^{\mathrm{Assy}(3)}(\gamma) \longmapsto s^{A(3)}(\gamma)\left(s^{A(3)}(s^{A(3)}(\gamma))\right)^{-1} = s^{A(3)}(\gamma)(s^{A(3)}(\gamma))^{-1} = 1 \right)$

$$0 \to H^1(\overline{D}_X, \triangle_\Theta \otimes \mathbb{Z}/\ell)$$

$$\ddot{\eta}^\Theta = s^{\mathrm{Assy}(3)}(G_K)$$

$$0 \to H^1(\overline{D}_X / s^{\mathrm{Assy}(3)}(G_K), \triangle_\Theta \otimes \mathbb{Z}/\ell) \to H^1(\Pi^{\mathrm{tp}}_{\ddot{Y}}, \triangle_\Theta \otimes \mathbb{Z}/\ell) \to H^1(\Pi^{\mathrm{tp}}_{\ddot{\underline{Y}}}, \triangle_\Theta \otimes \mathbb{Z}/\ell)$$

$$\ddot{\eta}^\Theta \longmapsto 0$$

(Recall)

$\rightsquigarrow$ Image of $\ddot{\eta}^\Theta \in H^1(\Pi^{\mathrm{tp}}_{\ddot{\underline{Y}}}, \triangle_\Theta)$ arises from $\ell \triangle_\Theta \subset \triangle_\Theta$

$\exists\ \ddot{\underline{\eta}}^\Theta \in H^1(\Pi^{\mathrm{tp}}_{\ddot{\underline{Y}}}, \ell\triangle_\Theta)$

well-def. up to $(\mathcal{O}_K^\times)^\ell$ ⟶ well-def up to $\mathcal{O}_K^\times$-mult

"ℓ-th root of the etale theta funct"

$\ddot{\underline{\eta}}^{\Theta, \ell\underline{\mathbb{Z}} \times \mu_2}$: $\Pi^{\mathrm{tp}}_{\underline{\underline{X}}} / \Pi^{\mathrm{tp}}_{\ddot{\underline{Y}}} \cong \ell\underline{\mathbb{Z}} \times \mu_2$-orbit of $\ddot{\underline{\eta}}^\Theta$

<u>Def 7.12</u> ([EtTh, Def 2.7]) We call $\ddot{\eta}^{\Theta, \ell\underline{\mathbb{Z}} \times \mu_2}$ <u>of standard type</u>

$\underset{\mathrm{def}}{\Leftrightarrow}$ $\ddot{\eta}^{\Theta, \underline{\mathbb{Z}} \times \mu_2}$: of std type

Cor 7.13 (

$\underline{X}$ (

Cor 7.13 (Const. Mult. Rig. of l-th Root of the Étale Theta Fct [EtTh, Cor 2.8])

$\underline{\underline{X}}$ (resp. $^{\dagger}\underline{\underline{X}}$) over K (resp. $^{\dagger}K$) $/\mathbb{Q}$ (fin) as before

$^{\dagger}(-)$ $\quad \gamma : \Pi^{\mathrm{tp}}_{\underline{\underline{X}}} \xrightarrow{\sim} \Pi^{\mathrm{tp}}_{^{\dagger}\underline{\underline{X}}}$ isom. of top. gps

(1). γ : preserves the property that $\ddot{\eta}^{\Theta, l\mathbb{Z} \times \mu_2}$ is of std type

this collection of classes up to a μ_l-mult. det's

(2). Assume cusps of $\underline{X}$ are rational $/K$, residue char. of $K \neq l$, $\mu_l \subset K$

$\Rightarrow$ $\{\pm 1\}$-str. of Prop 7.9 (3) det's μ_{2l}-str. at the decomp. gps of the cusps of $\underline{\underline{X}}$

$\ddot{\eta}^{\Theta}$ compatibly com. with the l preserved by γ.

Π Prop 7.9 & Cor 7.10

Cor 7.14 ([EtTh, Cor 2.9])

Assume $\mu_l \subset K$

cusps of $\underline{X}$ $\quad$ labelling $\quad$ labelling

nined. comp. $\quad$ by $\mathbb{Z}$

$$\{\text{Cusps of } \underline{\underline{X}}\} / \mathrm{Aut}_K(\underline{\underline{X}}) \xrightarrow{\sim} (\mathbb{Z}/l\mathbb{Z})/\{\pm 1\}$$

& preserved by $\forall \gamma : \Pi^{\mathrm{tp}}_{\underline{\underline{X}}} \xrightarrow{\sim} \Pi^{\mathrm{tp}}_{\underline{\underline{X}}}$ of top. gps.

$\underline{\underline{X}}$

μ_l

$\underline{X}$

ét $\quad$ ét

X

§7.4 Three Rigidities of Mono-Theta Environment

	mono-theta env. Θ	bi-theta env. $\Theta, \Theta_{\mathrm{bi}}$
cycl. rig.	delicately ○ quadraticity $[\ ,\ ]$	trivially ○
const. mult. rig.	delicately ○ ell. cusp'ation	trivially ○
discrete rig.	○	×

$\{\underline{q}^{j^2}\}_j \longrightarrow \underline{q}$ $\quad \underline{q}\,\Theta \hookleftarrow \underline{q}^{j^2}\,\vartheta$

$\rightsquigarrow \approx 0 \qquad -\log \underline{q} \qquad \rightsquigarrow 0 \leq -|\mathrm{ht}| + |\mathrm{ind\,et}|$

$(1+\epsilon)(\log\text{-diff} + \log\text{-cond})$

$\rightsquigarrow |\mathrm{ht}| \lesssim (1+\epsilon)(\cdots)$

$N > 1$

① $\{\underline{q}^{j^2}\} \longrightarrow \underline{q}^{N}$? $\overset{\lambda \to 0}{\approx} \ell$ $\quad \rightsquigarrow N \cdot 0 \leq -|\mathrm{ht}| + |\mathrm{ind\,et}|$

works

② $\{(\underline{q}^{j^2})^N\} \longrightarrow \underline{q}$? $\quad$ It worked $\quad \rightsquigarrow 0 \leq -N|\mathrm{ht}| + |\mathrm{ind\,et}|$

$\rightsquigarrow |\mathrm{ht}| \leq \frac{1}{N}(1+\epsilon)\left(\begin{matrix}\log\text{-diff.} \\ +\log\text{-cond.}\end{matrix}\right)$

does not work

$\rightsquigarrow$ contradiction
analytic number theory

$\mathbb{\Pi}^N \subset \prod_j$ $\quad$ const. mult. rig. $\quad [\ ,\ ]$ $\xrightarrow{\text{quadraticity}}$ mono-theta ~~cycl. rig~~

[EtTh, Rem 5.12.5] $\qquad$ [EtTh, Rem 2.19.3]

$\mathcal{L}^{\otimes N} \not\cong \mathcal{L} \rightsquigarrow (\ )^{\mathrm{LM}}$ $\quad \Theta^N \quad (\Theta)^N \quad N > 1$

Θ — cycl. rig.

calculation in HA

$\sum_j j^2 (\ ) \approx \frac{\ell^2}{4}(\ )$

$\|$

$\sum_j j (\ ) \approx \frac{\ell^2}{24}(\ )$

miracle equality

log-Kummer

[IUTchIII, Rem 2.3.3]

2,8])

lt.

$\subset K$

the cusps

of $\underline{X}$

labelling

labelled

Def. 7.15 $N \geq 1$ $\quad \Pi_{\mu_N,K} := \mu_N \rtimes G_K$

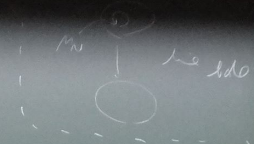
For a top. gp Π w/ surj. $\Pi \twoheadrightarrow G_K$

put $\Pi[\mu_N] := \Pi \times_{G_K} \Pi_{\mu_N,K}$, $\Delta[\mu_N] := \ker(\Pi[\mu_N] \twoheadrightarrow G_K)$

$= \Delta \times \mu_N$

$(\Delta := \ker(\Pi \twoheadrightarrow G_K))$

cyclotomic envelope of $\Pi \twoheadrightarrow G_K$



$\mu_N(\Pi[\mu_N]) := \ker(\Pi[\mu_N] \twoheadrightarrow \Pi)$

(mod N) cyclotome of the cycl. env. $\Pi[\mu_N]$

taut. sect. $G_K \longrightarrow \Pi_{\mu_N,K}$ of $\Pi_{\mu_N,K} \twoheadrightarrow G_K$

$\rightsquigarrow s^{\mathrm{alg}}_{\Pi} : \Pi \longrightarrow \Pi[\mu_N]$

mod N tautological section

For any obj w/ $\Pi[\mu_N]$-conj. action, we call a μ_N-orbit a μ_N-conj. class

Prop. 7.17 ([E+Th, Prop. 2.12])

(1). $\ker\left((\triangle^{\mathrm{top}}_{\underline{X}})^{\Theta} \to (\triangle^{\mathrm{top}}_{\underline{X}})^{\mathrm{ell}}\right) = l\triangle_{\Theta} \subset \left[(\triangle^{\mathrm{top}}_{\underline{X}})^{\Theta}, (\triangle^{\mathrm{top}}_{\underline{X}})^{\Theta}\right]$

we have an incl.

$s^{\Theta}, s^{\mathrm{alg}}$

(2). We have an equality

$$\left[(\triangle^{\mathrm{top}}_{\underline{X}})^{\Theta}[\mu_N], (\triangle^{\mathrm{top}}_{\underline{X}})^{\Theta}[\mu_N]\right] \cap (l\triangle_{\Theta})[\mu_N] = \mathrm{Im}\left(s^{\mathrm{alg}}_{(\triangle^{\mathrm{top}}_{\underline{X}})^{\Theta}}\Big|_{l\triangle_{\Theta}} : l\triangle_{\Theta} \to (\triangle^{\mathrm{top}}_{\underline{X}})^{\Theta}[\mu_N]\right)$$

$s^{\mathrm{alg}}_{(\triangle^{\mathrm{top}}_{\underline{X}})^{\Theta}}\Big|_{l\triangle_{\Theta}}$

restr. to $l\triangle_{\Theta}$

$$s^{\mathrm{alg}}_{(\triangle^{\mathrm{top}}_{\underline{X}})^{\Theta}} : (\triangle^{\mathrm{top}}_{\underline{X}})^{\Theta} \to (\triangle^{\mathrm{top}}_{\underline{X}})^{\Theta}[\mu_N]$$

mod N taut. section

☺ (1). str. of Heisenberg gp $(\triangle^{\mathrm{top}}_{X})^{\Theta}$

(2) $\Leftarrow$ (1)

Rm 7.17.1

$\underline{X}$

$\downarrow$

$\underline{X}$

does not work

$\mathrm{Im}(s^{\mathrm{alg}})$

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$s^{\mathrm{alg}}_{\underset{=}{\ddot{Y}}} : \Pi^{\mathrm{tp}}_{\underset{=}{\ddot{Y}}} \xrightarrow{s^{\mathrm{alg}}_{\Pi^{\mathrm{tp}}_{\underset{=}{\ddot{Y}}}}} \Pi^{\mathrm{tp}}_{\underset{=}{\ddot{Y}}}[\mu_N] \hookrightarrow \Pi^{\mathrm{tp}}_{\underset{=}{Y}}[\mu_N]$

the composite

mod N algebraic section

Take the composite $\eta : \Pi^{\mathrm{tp}}_{\underset{=}{\ddot{Y}}} \xrightarrow{\text{mod } N \text{ red.}} l\Delta_\Theta \otimes \mathbb{Z}/N \overset{\text{cycl. thery}}{\cong} \mu_N$

any elt in $\ddot{\eta}^{\Theta, l \underline{\mathbb{Z}} \times \mu_2} \subset H^1(\Pi^{\mathrm{tp}}_{\underset{=}{\ddot{Y}}}, l\Delta_\Theta)$

1-cocycle

Def 7.18 (Mono-Theta Environment [E

Put $s^{\Theta}_{\ddot{\underline{Y}}} := \ddot{\eta}^{-1} \cdot s^{\mathrm{alg}}_{\ddot{\underline{Y}}} : \Pi^{\mathrm{tp}}_{\ddot{\underline{Y}}} \longrightarrow \Pi^{\mathrm{tp}}_{\ddot{\underline{Y}}}[\mu_N]$

<u>mod N theta section</u>

Note $s^{\Theta}_{\ddot{\underline{Y}}}$: homomorphic.

the natural outer action

$\mathrm{Gal}(\underline{Y}/\underline{X}) \cong \Pi^{\mathrm{tp}}_{\underline{X}} / \Pi^{\mathrm{tp}}_{\underline{Y}} \cong \Pi^{\mathrm{tp}}_{\underline{X}}[\mu_N] / \Pi^{\mathrm{tp}}_{\underline{Y}}[\mu_N] \hookrightarrow \mathrm{Out}(\Pi^{\mathrm{tp}}_{\underline{Y}}[\mu_N])$

$\mathrm{Gal}(\underline{Y}/\underline{X}) \overset{\mathrm{out}}{\curvearrowright} \Pi^{\mathrm{tp}}_{\underline{Y}}[\mu_N]$ fixes $\mathrm{Im}(s^{\mathrm{alg}}_{\ddot{\underline{Y}}} : \Pi^{\mathrm{tp}}_{\ddot{\underline{Y}}} \to \Pi^{\mathrm{tp}}_{\ddot{\underline{Y}}}[\mu_N])$ up to conj. by μ_N

($s^{\mathrm{alg}}_{\ddot{\underline{Y}}}$ extends $\Pi^{\mathrm{tp}}_{\underline{X}} \to \Pi^{\mathrm{tp}}_{\underline{X}}[\mu_N]$),

$\rightsquigarrow$ $s^{\Theta}_{\ddot{\underline{Y}}}$ up to $\Pi^{\mathrm{tp}}_{\underline{X}}[\mu_N]$-conj. is indep. of the choice of an N-th root of $\ddot{\underline{\eta}}^{\Theta, \ell\mathbb{Z} \times \mu_2}$

we have natural outer action

$K^{\times} \twoheadrightarrow K^{\times}/(K^{\times})^N \cong H^1(G_K, \mu_N) \hookrightarrow H^1(\Pi^{\mathrm{tp}}_{\underline{Y}}, \mu_N) \longrightarrow \mathrm{Out}(\Pi^{\mathrm{tp}}_{\underline{Y}}[\mu_N])$

$s \longmapsto \left(s^{\mathrm{alg}}_{\Pi^{\mathrm{tp}}_{\underline{Y}}}(g) a \mapsto s(g) s^{\mathrm{alg}}_{\Pi^{\mathrm{tp}}_{\underline{Y}}}(g) \cdot a, \quad g \in \Pi^{\mathrm{tp}}_{\underline{Y}},\ a \in \mu_N \right)$

Prop. 7.1

(1)

(2)

[(

Note a

Prop. 7.17 ([E+Th, Prop. 2.12])

(1). $\ker\left((\Delta^{+tp}_{\underline{X}})^{\Theta} \to (\Delta^{+tp}_{\underline{X}})^{ell} \right) = l\Delta_{\Theta} \subseteq \left[(\Delta^{+tp}_{\underline{X}})^{\Theta}, (\Delta^{+tp}_{\underline{X}})^{\Theta} \right]$

we have an incl.

(2). We have an equality

$$\left[(\Delta^{+tp}_{\underline{X}})^{\Theta}[\mu_N], (\Delta^{+tp}_{\underline{X}})^{\Theta}[\mu_N] \right] \cap (l\Delta_{\Theta})[\mu_N]$$
$$= \mathrm{Im}\left(s^{alg}_{(\Delta^{+tp}_{\underline{X}})^{\Theta}}\Big|_{l\Delta_{\Theta}} : l\Delta_{\Theta} \to (\Delta^{+tp}_{\underline{X}})^{\Theta}[\mu_N] \right)$$

s^{Θ}, s^{alg}

Note also

$\forall$ elt $\in \mathrm{Im}(K^{\times}) := \mathrm{Im}\left(K^{\times} \to \mathrm{Out}(\Pi^{tp}_{\underline{Y}}[\mu_N])\right)$

lift $\exists\sigma$ an elt $\in \mathrm{Aut}(\Pi^{+tp}_{\underline{Y}}[\mu_N])$

which induces the id autom. of both the quot.

$\Pi^{tp}_{\underline{Y}}[\mu_N] \to \Pi^{tp}_{\underline{Y}}$ & the kernel of this quot.

Def 1.10 (Mono-Theta Environment [EtTh, Def. 2.13])

$$\mathcal{D}_{\underline{Y}} := \langle \mathrm{Im}(K^\times), \underbrace{\mathrm{Gal}(\underline{Y}/\underline{X})}_{\cong \, l\underline{\mathbb{Z}}} \rangle \subset \mathrm{Out}(\Pi^{tp}_{\underline{\ddot{Y}}}[\mu_N])$$

(1) We call the following collection of data a mod N model mono-theta environment:

- the top. gp $\Pi^{tp}_{\underline{\ddot{Y}}}[\mu_N]$
- the subgp $\mathcal{D}_{\underline{Y}}$ $(\subset \mathrm{Out}(\Pi^{tp}_{\underline{\ddot{Y}}}[\mu_N]))$, and
- the μ_N-conj. class of subgps in $\Pi^{tp}_{\underline{\ddot{Y}}}[\mu_N]$ det'd by the image of the theta section $s^{\Theta}_{\underline{\ddot{Y}}}$

(2) We call any collection $\mathbb{M} = (\Pi, \mathcal{D}_\Pi, s^{\Theta}_\Pi)$ of the following data a mod N mono-theta environment

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...[2.13])

... $\subset \mathrm{Out}(\Pi^{tp}_{\xi}[\mu_N])$

mod N model

... and

... $^{tp}_{\xi}[\mu_N]$

... theta section s^{Θ}_{ξ}

... $\mathcal{D}_\Pi, s^{\Theta}_{\Pi})$ of the following

... environment

- top. gp Π
- a subgp $\mathcal{D}_\Pi \subset \mathrm{Out}(\Pi)$
- a collection of subgps s^{Θ}_{Π} of Π

s.t. $\Pi \overset{\exists}{\simeq} \Pi^{tp}_{\xi}[\mu_N]$ of top. gps which maps $\mathcal{D}_\Pi \subset \mathrm{Out}(\Pi)$ to $\mathcal{D}_\xi$ & s^{Θ}_{Π} to the μ_N-conj. class of subgps in $\Pi^{tp}_{\xi}[\mu_N]$ det'd by the image of s^{Θ}_{ξ}

(3). $\mathbb{M} = (\Pi, \mathcal{D}_\Pi, s^{\Theta}_{\Pi})$, $^{\dagger}\mathbb{M} = (^{\dagger}\Pi, \mathcal{D}_{^{\dagger}\Pi}, s^{\Theta}_{^{\dagger}\Pi})$: mono-theta env.

isom of mod N mono-theta env. $\mathbb{M} \simeq {}^{\dagger}\mathbb{M}$

to be an isom. of top. gps $\Pi \simeq {}^{\dagger}\Pi$

which maps $\mathcal{D}_\Pi$ to $\mathcal{D}_{^{\dagger}\Pi}$, s^{Θ}_{Π} to $s^{\Theta}_{^{\dagger}\Pi}$,

$\mathbb{M}$: mod N mono-theta env. $\mathbb{M} = (\Pi, \mathcal{D}_\Pi, s^{\Theta}_{\Pi})$

$^{\dagger}\mathbb{M}$: mod M : "

$M | N$ a hom. of mono-theta env. $\mathbb{M} \to {}^{\dagger}\mathbb{M}$

to be an isom $\underbrace{\mathbb{M}_M}_{\text{mod } M \text{ mono-theta env. induced by } \mathbb{M}} \simeq {}^{\dagger}\mathbb{M}$

Rem 7.18.1 We can consider mod N bi-theta env $\mathbb{B} = (\Pi, \mathcal{D}_\Pi, s^{\Theta}_{\Pi}, s^{\mathcal{D}_\Pi}_{\Pi})$

Lem 7.19 ([EtTh, Prop 2.14])

(1). We have the following gp-thic characterization of the image of the taut. section of $(l\Delta_\Theta)[\mu_N] \to l\Delta_\Theta$ as the following subgp of $(\Delta^{\text{temp}}_{\underline{Y}})^{\Theta}[\mu_N]$:

$$(l\Delta_\Theta)[\mu_N] \cap \left\{ \gamma(a)a^{-1} \in (\Delta^{\text{temp}}_{\underline{Y}})^{\Theta}[\mu_N] \;\middle|\; \begin{array}{l} a \in (\Delta^{\text{temp}}_{\underline{Y}})^{\Theta}[\mu_N], \\ \gamma \in \mathrm{Aut}(\Pi^{\text{temp}}_{\underline{Y}}[\mu_N]) \text{ s.t. } (*) \end{array} \right\}$$

$(*)$: the image of γ in $\mathrm{Out}(\Pi^{\text{temp}}_{\underline{Y}}[\mu_N])$ belongs to $\mathcal{D}_{\underline{Y}}$, and γ induces the identity on the quot. $\Pi^{\text{temp}}_{\underline{Y}}[\mu_N] \to \Pi^{\text{temp}}_{\underline{Y}} \to G_K$

(2). $t^{\Theta}_{\underline{Y}} : \Pi^{\text{temp}}_{\underline{Y}} \to \Pi^{\text{temp}}_{\underline{Y}}[\mu_N]$ a section obtained as a conj. of $s^{\Theta}_{\underline{Y}}$ rel. to the actions of $K^\times$ and $l\mathbb{Z}$

Put $\delta := (s^{\Theta}_{\underline{Y}})^{-1} t^{\Theta}_{\underline{Y}}$: 1-cocycle of $\Pi^{\text{temp}}_{\underline{Y}}$ valued in μ_N

$\alpha_\delta \in \mathrm{Aut}(\Pi^{\text{temp}}_{\underline{Y}}[\mu_N])$: the autom. given by $s^{\mathcal{D}_Y}_{\Pi^{\text{temp}}_{\underline{Y}}}(g)a \mapsto \delta(g)s^{\mathcal{D}_Y}_{\Pi^{\text{temp}}_{\underline{Y}}}(g)a$ $\quad (g \in \Pi^{\text{temp}}_{\underline{Y}},\ a \in \mu_N)$

which induces the id aut. on both the quot.

$\Pi^{\text{temp}}_{\underline{Y}}[\mu_N] \to \Pi^{\text{temp}}_{\underline{Y}}$ & the kernel of this quot.

Then $\tilde{\alpha}_\delta$ extends to an autom. $\alpha_\delta \in \mathrm{Aut}(\Pi^{tp}_{\underline{Y}}[\mu_N])$
which induces an id aut on both the quot. $\Pi^{tp}_{\underline{Y}}[\mu_N] \to \Pi^{\wedge}_{\underline{Y}}$.
The conj by α_δ maps $s^{\Theta}_{\underline{Y}}$ to $t^{\Theta}_{\underline{Y}}$ & its kernel.
& preserves the subgp $\mathcal{D}_{\underline{Y}}$ $(\subset \mathrm{Out}(\Pi^{tp}_{\underline{Y}}[\mu_N]))$

(3). $\mathbb{M} = (\Pi^{tp}_{\underline{Y}}[\mu_N], \mathcal{D}_{\underline{Y}}, s^{\Theta}_{\underline{Y}})$: the mod N model mono-theta env.

$\forall$ autom. of $\mathbb{M}$ induces an autom of $\Pi^{tp}_{\underline{Y}}$ (by Lem 7.16(2))

$\rightsquigarrow$ autom. of $\Pi^{tp}_{\underline{X}} = \mathrm{Aut}(\Pi^{tp}_{\underline{Y}}) \overset{\mathrm{out}}{\rtimes} \mathrm{Im}(\mathcal{D}_{\underline{Y}} \to \mathrm{Out}(\Pi^{tp}_{\underline{Y}}))$

$= \mathrm{Aut}(\Pi^{tp}_{\underline{Y}}) \times_{\mathrm{Out}(\Pi^{tp}_{\underline{Y}})} \mathrm{Im}(\mathcal{D}_{\underline{Y}} \to \mathrm{Out}(\Pi^{tp}_{\underline{Y}}))$

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also induces $\rightsquigarrow$ an autom. of the set of cusps of $\underline{Y}$

wl. to the labelling by $\mathbb{Z}$ on these cusps,

this induces an autom of $\mathbb{Z}$ given by $l\mathbb{Z} \rtimes \{\pm 1\}$

$\rightsquigarrow \mathrm{Aut}(\mathbb{M}) \to l\mathbb{Z} \rtimes \{\pm 1\}$

☺ (1). Take a lift $\gamma \in \mathrm{Aut}((\Pi^{tp}_{\underline{Y}})[\mu_N])$ of an elt $\in \mathrm{Im}(K^\times) \subset \mathcal{D}_{\underline{Y}}$
s.t. satisfies ⊛

$\gamma = \gamma_1 \gamma_2$ $\quad \gamma_1 \in \mathrm{Inn}(\Pi^{tp}_{\underline{Y}}[\mu_N])$, $\quad \gamma_2 \in \mathrm{Aut}(\Pi^{tp}_{\underline{Y}}[\mu_N])$

the img of γ_2 in $\mathrm{Out}(\Pi^{tp}_{\underline{Y}}[\mu_N])$ is in $\mathrm{Im}\begin{pmatrix} K^\times \to H^1(G_K, \mu_N) \\ \to H^1(\Pi^{tp}_{\underline{Y}}, \mu_N) \\ \to \mathrm{Out}(\Pi^{tp}_{\underline{Y}}[\mu_N]) \end{pmatrix}$

& the autom induced by γ_2 on the quot.
$\Pi^{tp}_{\underline{Y}}[\mu_N] \to \Pi^{tp}_{\underline{Y}}$ & its ker are trivial

$H^1(G_K, \mu_N) \to H^1(\Pi^{tp}_{\underline{Y}}, \mu_N) \to H^1(\Delta^{tp}_{\underline{Y}}, \mu_N)$

$\searrow \mathrm{Out}(\Delta^{tp}_{\underline{Y}}(\mu_N))$

$\rightsquigarrow \gamma_2 \curvearrowright \Delta^{tp}_{\underline{Y}}(\mu_N)$ is inner

On the other hand, $\gamma_1 \curvearrowright G_K$ trivial ($\because$ $\gamma_2 \curvearrowright G_K$ trivial & condition ⊛)

G_K: center-free (Prop 2.7 (1c)) is in $\subset \mathrm{Inn}(\Delta^{tp}_{\underline{Y}}(\mu_N))$

$\rightsquigarrow \gamma_1 \in \mathrm{Inn}(\Pi^{tp}_{\underline{Y}}(\mu_N))$

$\rightsquigarrow \gamma = \gamma_1\gamma_2 \curvearrowright \Delta^{tp}_{\underline{Y}}(\mu_N)$ is also inner $\rightsquigarrow$ the inner automorphism induced by γ on $(\Delta^{tp}_{\underline{Y}})^{\Theta}(\mu_N)$ is trivial (1)

$\left((\Delta^{tp}_{\underline{Y}})^{\Theta}(\mu_N) \cong \ell\underline{\mathbb{Z}} \times \hat{\mathbb{Z}}(1) \times \mu_N\right)$ abelian, Prop 7.1 (2)

(2): omit

(3) $\Leftarrow$ (2) //

<u>Cor 7.20</u> (Group-Theoretic Reconstruction of Mono-Theta Environment) [EtTh, Cor 2.18]

$N \geq 1$, ℓ: prime, $\underline{X}$ smooth log-curve of type $(1, (\mathbb{Z}/\ell\mathbb{Z})^{\Theta})$ / K fin. $\mathbb{Q}_p$

$\ell, p > 2$, $\ddot{K} = K$.

$\ddot{U}^2 \quad \ddot{\underline{Y}}$

$\underline{Y}$

$\mathbb{M}_N$: the mod N model mono-theta env.

(1). ${}^{\dagger}\Pi^{tp}_{\underline{X}}$ top. gp. $\cong \Pi^{tp}_{\underline{X}}$

$\rightsquigarrow$ subquot. ${}^{\dagger}\Pi^{tp}_{\underline{Y}}$, ${}^{\dagger}\Pi^{tp}_{\ddot{\underline{Y}}}$, ${}^{\dagger}G_K$, ${}^{\dagger}(\ell\Delta_\Theta)$, ${}^{\dagger}(\Delta^{tp}_{\underline{X}})^{\Theta}$, ${}^{\dagger}(\Pi^{tp}_{\underline{X}})^{\Theta}$,

${}^{\dagger}(\Delta^{tp}_{\underline{Y}})^{\Theta}$, ${}^{\dagger}(\Pi^{tp}_{\underline{Y}})^{\Theta}$ of ${}^{\dagger}\Pi^{tp}_{\underline{X}}$

$\overbrace{\quad}$ gp th'c alg'm.

and • a collection of subgps of $^\dagger\Pi^{\mathrm{tp}}_{\underline{\underline{X}}}$ for each elt $(\mathbb{Z}/\ell\mathbb{Z})/\{\pm1\}$
s.t. any isom. $^\dagger\Pi^{\mathrm{tp}}_{\underline{\underline{X}}} \xrightarrow{\sim} \Pi^{\mathrm{tp}}_{\underline{\underline{X}}}$ maps

(2) $\Pi \mapsto \mathcal{M}$
(3) $\mathcal{M} \mapsto \Pi$

• the above subquots to the subquots
$\Pi^{\mathrm{tp}}_{\underline{\underline{Y}}}$, $\Pi^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}}$, G_K, $\ell\Delta_\Theta$, $(\Delta^{\mathrm{tp}}_{\underline{\underline{X}}})^\Theta$, $(\Pi^{\mathrm{tp}}_{\underline{\underline{Y}}})^\Theta$, $(\Delta^{\mathrm{tp}}_{\underline{\underline{Y}}})^\Theta$, $(\Pi^{\mathrm{tp}}_{\underline{\underline{Y}}})^\Theta$
of $\Pi^{\mathrm{tp}}_{\underline{\underline{X}}}$ resp'ly

and • the above collection of subgps to the collection of
cuspidal decomp. gps of $\Pi^{\mathrm{tp}}_{\underline{\underline{X}}}$ det'd by the label in $(\mathbb{Z}/\ell)/\{\pm1\}$
in a functorial manner w.r.t. isoms of top. gps.
(no need of any reference isom. to $\Pi^{\mathrm{tp}}_{\underline{\underline{X}}}$)

---

(2). $\exists$ gp th'c alg'm
$$(\text{“}\Pi \mapsto \mathcal{M}\text{”}) \quad {}^\dagger\Pi^{\mathrm{tp}}_{\underline{\underline{X}}} \longmapsto {}^\dagger\mathcal{M} = ({}^\dagger\Pi, \mathcal{O}_{{}^\dagger\Pi}, \mathfrak{s}^\Theta_{{}^\dagger\Pi})$$
where $^\dagger\Pi := {}^\dagger\Pi^{\mathrm{tp}}_{\underline{\underline{Y}}} \times_{{}^\dagger G_K} (\ell\,{}^\dagger\Delta_\Theta \otimes \mathbb{Z}/N) \rtimes {}^\dagger G_K)$
up to isom in a functorial manner w.r.t.
isom. of top. gps (no need of any ref. isom. to $\Pi^{\mathrm{tp}}_{\underline{\underline{X}}}$)
(See also [EtTh, Cor 2.18(ii)] for a stronger form)

$\mathcal{O} \in$ condition $\circledast$)
$\in \mathrm{Inn}(\Delta^{\mathrm{tp}}_{\underline{\underline{Y}}}(p_N))$
the inner autom'd by Y
$(\Delta^{\mathrm{tp}}_{\underline{\underline{Y}}})^\Theta(p_N)$ ... (1)
$\ddot{U}^2$ $\underline{\ddot{Y}}$
$\underline{Y}$
$(\Delta^{\mathrm{tp}}_{\underline{\underline{X}}})^\Theta$, $^\dagger(\Pi^{\mathrm{tp}}_{\underline{\underline{X}}})^\Theta$,
$)^\Theta$ of $^\dagger\Pi^{\mathrm{tp}}_{\underline{\underline{Y}}}$

(3)

"$\mathbb{M} \mapsto \Pi$"

${}^{\dagger}\mathbb{M} = ({}^{\dagger}\Pi, \mathcal{D}_{{}^{\dagger}\Pi}, s^{\Theta}_{{}^{\dagger}\Pi})$ a mod N mono-theta environ. $\cong \mathbb{M}_N$

gp th'c alg'm $\quad {}^{\dagger}\Pi \underset{\text{quot.}}{\longrightarrow} {}^{\dagger}\Pi^{\text{tp}}_{\underline{Y}}$ s.t. any isom ${}^{\dagger}\mathbb{M} \xrightarrow{\sim} \mathbb{M}_N$

maps this quot. to the quot.

$\Pi^{\text{tp}}_{\underline{Y}}(\mu_N) \to \Pi^{\text{tp}}_{\underline{Y}}$

in a functorial manner w.r.t. isoms of mono-theta env.

(no need of any ref. isom to $\mathbb{M}_N$)

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Furthermore, any isom ${}^{\dagger}\mathbb{M} \xrightarrow{\sim} \mathbb{M}_N$ induces an isom

$${}^{\dagger}\Pi^{\text{tp}}_{\underline{X}} := \mathrm{Aut}({}^{\dagger}\Pi^{\text{tp}}_{\underline{Y}}) \times_{\mathrm{Out}({}^{\dagger}\Pi^{\text{tp}}_{\underline{Y}})} \mathrm{Im}(\mathcal{D}_{{}^{\dagger}\Pi} \to \mathrm{Out}({}^{\dagger}\Pi^{\text{tp}}_{\underline{Y}}))$$

$$\xrightarrow{\sim} \Pi^{\text{tp}}_{\underline{X}} \qquad \{1\} \subset {}^{\dagger}\Pi^{\text{tp}}_{\underline{X}}$$

top. gr't'd by taking ${}^{\dagger}\Pi^{\text{tp}}_{\underline{Y}} \xrightarrow{\sim} \mathrm{Aut}(\ :\ ) \times_{\mathrm{Out}(:)}$ to be an open subgp.

$$\begin{array}{ccc} & \mathrm{id} & \\ \Pi & & \Pi \\ (1) & & (3) \\ & \mathbb{M} & \end{array}$$

If we apply alg'm of (2) to ${}^{\dagger}\Pi^{\text{tp}}_{\underline{Y}}$ $\longmapsto$ the resulting mono-theta env. $\cong$ the original ${}^{\dagger}\mathbb{M}$

via an isom which induces the id. on ${}^{\dagger}\Pi^{\text{tp}}_{\underline{Y}}$

(4). $^{\dagger}\mathcal{M} = ({}^{\dagger}\Pi, \mathcal{D}_{{}^{\dagger}\Pi}, \mathfrak{s}^{\Theta}_{{}^{\dagger}\Pi})$, $^{\ddagger}\mathcal{M} = ({}^{\ddagger}\Pi, \mathcal{D}_{{}^{\ddagger}\Pi}, \mathfrak{s}^{\Theta}_{{}^{\ddagger}\Pi})$ mod N mono-theta env.

$\downarrow(3)$ $\qquad\qquad$ $\downarrow(3)$

${}^{\dagger}\Pi^{\mathrm{tmp}}_{\underline{\underline{X}}}$ $\qquad\qquad$ ${}^{\ddagger}\Pi^{\mathrm{tmp}}_{\underline{\underline{X}}}$

$$\rightsquigarrow \mathrm{Isom}^{\mu_N\text{-conj}}({}^{\dagger}\mathcal{M}, {}^{\ddagger}\mathcal{M}) \longrightarrow \mathrm{Isom}({}^{\dagger}\Pi^{\mathrm{tmp}}_{\underline{\underline{X}}}, {}^{\ddagger}\Pi^{\mathrm{tmp}}_{\underline{\underline{X}}})$$

(*)

the set of μ_N-conj. classes of isom's

surj w/ fibers of card. $=1$ (resp. $=2$)

if N is odd (resp. even)

In particular, $^{\forall}M|N$, $\mathrm{Aut}^{\mu_N\text{-conj}}({}^{\dagger}\mathcal{M}) \longrightarrow \mathrm{Aut}^{\mu_M\text{-conj}}({}^{\dagger}\mathcal{M}_M)$

mod M mono-theta env. det'd by ${}^{\dagger}\mathcal{M}$

$\#$ker $\&$ $\#$coker (≤ 2) $= \#$ker $\&$ $\#$coker $\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}/N\mathbb{Z}) \to \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}/M\mathbb{Z})$ resp'ly

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(∵) (1). ${}^{\dagger}\Pi^{\mathrm{tmp}}_{\underline{\underline{X}}} \twoheadrightarrow {}^{\dagger}G_K$ by Lem 6.2.

other grnt's by Lem 7.8, Lem 7.10 & def's

labels of cusp. decomp gps by Lem 7.14

(2). $\Leftarrow$ def's

(3). By Lem 7.16(2) ${}^{\dagger}\Pi \twoheadrightarrow {}^{\dagger}\Pi^{\mathrm{tmp}}_{\underline{\underline{Y}}}$

${}^{\dagger}\Pi^{\mathrm{tmp}}_{\underline{\underline{X}}}$ def & temp-ness of ${}^{\dagger}\Pi^{\mathrm{tmp}}_{\underline{\underline{X}}}$ (Prop. 6.4)

The last claim $\Leftarrow$ def's, description of alg'm of (2)

(4). omit //

Th 7.21 (Three Rigidities of Mono-Theta Environment, [EtTh, Cor 2.19])

$N \geq 1$, l: prime, $\underline{X}$: smooth log-curve of type $(1, (\mathbb{Z}/l\mathbb{Z})^{\Theta})$ over K fin. $/\mathbb{Q}_p$

$l, p > 2$, $\ddot{K} = K$, $\mathbb{M}_N$: the mod N model mono-theta env.

(1) (Cycl. Rig.) $^{\dagger}\mathbb{M} = ({}^{\dagger}\Pi, \mathcal{D}_{{}^{\dagger}\Pi}, s^{\Theta}_{{}^{\dagger}\Pi})$ a mod N mono-theta env. $\cong \mathbb{M}_N$

${}^{\dagger}\Pi^{\mathrm{tmp}}_{\underline{X}}$ [Cor 7.20(3)]

$\exists$ gp th'c $\longrightarrow$ subquots ${}^{\dagger}(l\Delta_{\Theta}[\mu_N]) \subset {}^{\dagger}((\Delta^{\mathrm{tmp}}_{\underline{Y}})^{\Theta}[\mu_N]) \subset {}^{\dagger}((\Pi^{\mathrm{tmp}}_{\underline{Y}})^{\Theta}[\mu_N])$ of ${}^{\dagger}\Pi$ s.t.

any isom. $^{\dagger}\mathbb{M} \xrightarrow{\sim} \mathbb{M}_N$ maps these subquot's

to the subquots $l\Delta_{\Theta}[\mu_N] \subset (\Delta^{\mathrm{tmp}}_{\underline{Y}})^{\Theta}[\mu_N] \subset (\Pi^{\mathrm{tmp}}_{\underline{Y}})^{\Theta}[\mu_N]$ of $\Pi^{\mathrm{tmp}}_{\underline{Y}}[\mu_N]$

in a functorial manner w.r.t. isom's of mono-theta env. (no need of any ref. isom. to $\mathbb{M}_N$).

Moreover, $\exists$ gp th'c alg'm

$\longrightarrow$ two splittings of the nat. surj. ${}^{\dagger}(l\Delta_{\Theta}[\mu_N]) \twoheadrightarrow {}^{\dagger}(l\Delta_{\Theta})$

s.t. any isom. $^{\dagger}\mathbb{M} \xrightarrow{\sim} \mathbb{M}_N$ maps these two splittings

to the two splittings of the surj. $l\Delta_{\Theta}[\mu_N] \twoheadrightarrow l\Delta_{\Theta}$

det'd by the mod N alg section $s^{\mathrm{alg}}_{\underline{Y}}$ &

the mod N theta section $s^{\Theta}_{\underline{Y}}$

in a functorial manner w.r.t. isom's of mono-theta env. (no need of any ref. isom. to $\mathbb{M}_N$)

In particular, by taking the difference of these two splittings, there exist a gp-th'c alg'm

$\longrightarrow$ isom. of cyclotomes

(Cyc. Rig. Mono-Th.) $\quad {}^{\dagger}|l\Delta_\Theta| \otimes \mathbb{Z}/N\mathbb{Z} \xrightarrow{\sim} \mu_N({}^{\dagger}(l\Delta_\Theta[\mu_N]))$

s.t. any isom ${}^{\dagger}\mathbb{M} \xrightarrow{\sim} \mathbb{M}_N$ maps this isom of cyclotomes to the nat. isom. of cyclotomes $l\Delta_\Theta \otimes \mathbb{Z}/N\mathbb{Z} \xrightarrow{\sim} \mu_N(l\Delta_\Theta[\mu_N])$ in a functorial manner

w.r.t. isom's of mono-theta env.

(no need of any isom to ref. model $\mathbb{M}_N$)

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(2). (Discrete Rigidity)

any proj. system $({}^{\dagger}\mathbb{M}_N)_{N \geq 1}$ of mono-theta env.

is isom. to the nat. proj. system of the model mono-theta env. $(\mathbb{M}_N)_{N \geq 1}$

(3). (Constant Multiple Rigidity)

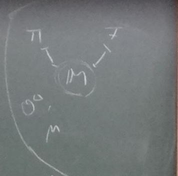
Assume that $\ddot{\eta}^{\Theta, l\mathbb{Z} \times \mu_2}$ is of std type.

$\exists$ proj. system of mono-theta env.

$({}^{\dagger}\mathbb{M}_N)_{N \geq 1} \underset{\text{gp-th'c alg'm}}{\longmapsto}$ a collection of classes of $H^1({}^{\dagger}\Pi^{\text{tp}}_{\underline{Y}}, {}^{\dagger}(l\Delta_\Theta))$

s.t. any isom $({}^{\dagger}\mathbb{M}_N)_{N \geq 1} \xrightarrow{\sim} (\mathbb{M}_N)_{N \geq 1}$

maps the above collection of classes to some multiple of the collection of classes $\ddot{\eta}^{\Theta, l\mathbb{Z} \times \mu_2}$



Rotaner $\Pi\sigma$

$\mu_N(l\Delta_\Theta[\mu_N])$

... to ref. model (M_N)

... theta env.

... env. $(M_N)_{N\geq 1}$

... tay ... $\dagger(l\Delta_\Theta))$

$\Pi\sigma$ ... $\Theta, \mathcal{O}^\times \cdot \mu_2$

---

by an elt of μ_Θ in a tautol manner w.r.t. isom. of proj. systems of mono-theta env.'s

(no need of any ref. isom. to $(M_N)_{N\geq 1}$)

proof) (1). $s^{\mathrm{alg}}_{\underline{\underline{Y}}}\Big|_{\mathrm{Ker}(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}} \to (\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}})^\Theta)} = s^\Theta_{\underline{\underline{Y}}}\Big|_{\underbrace{\mathrm{Ker}(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}} \to (\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}})^\Theta)}_{\text{can. reconst.}}}$ by Rem 7.2.1 (1).

We can also reconst. $\dagger(l\Delta_\Theta) \subset \dagger((\Delta^{\mathrm{temp}}_{\underline{\underline{Y}}})^\Theta) \subset \dagger((\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}})^\Theta)$ by Cor 7.20 (1) (3)

as the subset of (any μ_N-conj. class of) $s^\Theta_{\dagger\Pi}$ whose elts proj. to $\mathrm{Ker}(\dagger(\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}}) \to \dagger((\Pi^{\mathrm{temp}}_{\underline{\underline{Y}}})^\Theta))$

(Lem 7.16 (2), Cor 7.20 (3), Cor 7.20 (1))

We can recon. the splittings of the natural surj. $\dagger(l\Delta_\Theta[\mu_N]) \twoheadrightarrow \dagger(l\Delta_\Theta)$ given by

the theta section directly as $s^\Theta_{\dagger\Pi}$,

the alg. section by the alg'm of Lem 7.19 (1).

(2) follows from Lemma 7.20 (4), since $R^1\varprojlim \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}/N\mathbb{Z}) = 0$

$R^1\varprojlim_N \mu_N = 0$

(3). $\Leftarrow$ Lem 7.19 (3), Cor 7.13 & the cycl. rig (1) & the discrete rig (2). //

Rem 7.21.1 mono-theta vs bi-theta

(1) (cycl. rig.) mono-theta $\xLeftarrow[\text{delicate}]{}$ $[\ ,\ ]$

bi-theta $\xLeftarrow[\text{trivial}]{}$ s^{alg} is included in the data

(2) (Const. mult. rig.) mono-theta $\xLeftarrow[\text{delicate}]{}$ ell. cusp.

bi-theta $\xLeftarrow[\text{trivial}]{}$ the ratio ($\leftarrow$ étale theta class) is indep. of simultaneous const. mult. on s^{Θ}, s^{alg}

(3) (Discrete rig)

mono-theta has "shifting auto" $\ddot{\alpha}_{\delta}$ in Lem 7.19(2)

no base pt rel. to $\underline{\underline{Y}} \curvearrowleft l\underline{\mathbb{Z}}$ $(\subset \mathcal{D}_{\underline{Y}})$ ...

N nums $\rightsquigarrow$ $l\underline{\mathbb{Z}}$-torsors $\xrightarrow{\quad}$ $\varprojlim_N l\underline{\mathbb{Z}}/l\underline{\mathbb{Z}} = 0$-torsor

bi-theta N num $\rightsquigarrow$ $Nl\underline{\mathbb{Z}}$-torsors $\rightsquigarrow$ $\widehat{\mathbb{Z}}/\mathbb{Z}$-torsor

depends on N

$$\begin{cases} \text{bi} & 0 \to Nl\mathbb{Z} \to l\mathbb{Z} \to l\mathbb{Z}/Nl\mathbb{Z} \to 0 \\ \text{mono} & 0 \to l\mathbb{Z} \to l\mathbb{Z} \to l\mathbb{Z}/l\mathbb{Z} \to 0 \end{cases}$$

$\pi(M_N)_{N\geq 1}$

$\Pi^{+}_{\underline{Y}}|^{\Theta}|)$

3), cf 7.20(1)

al

7.19(1).

$/N\mathbb{Z}| = 0$

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lim (mono)

$$0 \to \overbrace{\varprojlim_{n} N\ell\mathbb{Z}}^{0} \to \ell\mathbb{Z} \to \ell\hat{\mathbb{Z}} \to \overbrace{R^1\varprojlim_{n} N\ell\mathbb{Z}}^{\ell\hat{\mathbb{Z}}/\ell\mathbb{Z}} \to 0$$

$$0 \to \underbrace{\varprojlim_{n} \ell\mathbb{Z}}_{\ell\mathbb{Z}} \xrightarrow{\sim} \ell\mathbb{Z} \to 0 \to \underbrace{R^1\varprojlim_{n} \ell\mathbb{Z}}_{0} \to 0$$

discrete rig. mono-theta

Rem 7.21.2 $\ddot{\Theta}^N$ $(N>1)$ does not work ~~cycl. rig~~, ~~const. mult.~~ rig.

Rem 7.21.3 three rigidities $\longleftrightarrow$ rtn. of Heisenberg gp?

$H^1(\Pi, -) \ni \log(\ddot{\Theta})$

affine transl. $\underset{\sim}{\hat{\mathbb{Z}}^\times} \cdot \log(\ddot{\Theta}) + \underset{\sim}{\hat{\mathbb{Z}}}$

cycl. rig. rigidifies $(\Delta_X^{\Theta})^{\Theta}$

const. mult. rig. rigidifies

disc rig. rigidifies $\hat{\mathbb{Z}}^\times \cong \mathrm{Aut}(\hat{\mathbb{Z}}(1))$

$$\begin{pmatrix} \text{cycl. rig.} & \text{const. mult. rig.} \\ 0 & \text{dis. rig.} \end{pmatrix}$$

(p-adic) analogue of KS-map

$\mathcal{O}$

$\mathcal{O}_{\Delta_\Theta}$

§7.5 From Frobenioids to Mono-Theta Environments

§7.5.1 a little bit on a review of Fr'd

feeling Galois cat. (Anabelioid) : Fundamental Groups (coverings)

$=$ Frobenioid : line bdles (divisors) on coverings

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Def 7.22 ([FrdI, Def 1.1])

(1). $\mathcal{D}$: cat.
contravariant fctor $\Phi : \mathcal{D} \longrightarrow$ (comm. Monoids)
is a monoid over $\mathcal{D}$
($\mathcal{D}^{\Phi}$) omit

(2). Φ : monoid over $\mathcal{D}$
cat. $\mathbb{F}_{\Phi}$: elementary Frobenioid assoc. to Φ

Obj' $= \mathrm{Obj}($
Morph $A, B \in$
$\phi_{\mathcal{D}} : A$
$z_{\phi} \in$
$\mathrm{Base}(\phi) :=$

...ironments

Groups

rings

...orm.

...monoids)

assoc. to Φ

$\mathrm{Obj}' = \mathrm{Obj}(\mathcal{D})$

Morph $A, B \in \mathrm{Obj}(\mathcal{D})$, $\phi : A \to B$

$$\phi = (\phi_{\mathcal{D}}, Z_\phi, n_\phi)$$

$$\begin{cases} \phi_{\mathcal{D}} : A \to B \text{ morph in } \mathcal{D} \\ Z_\phi \in \Phi(A),\ n_\phi \in \mathbb{N}_{\geq 1} \end{cases}$$

$$\mathrm{Base}(\phi) := \phi_{\mathcal{D}}, \quad \mathrm{Div}(\phi) := Z_\phi, \quad \deg_{Fr}(\phi) := n_\phi$$

$$A_{\mathcal{D}} := A, \quad B_{\mathcal{D}} := B$$

Composition $\phi = (\phi_{\mathcal{D}}, Z_\phi, n_\phi) : A \to B$

$\psi = (\psi_{\mathcal{D}}, Z_\psi, n_\psi) : B \to C$

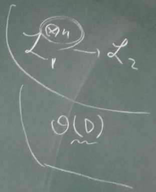
$$\psi \circ \phi := (\psi_{\mathcal{D}} \circ \phi_{\mathcal{D}},\ \phi_{\mathcal{D}}^*(Z_\psi) + n_\psi Z_\phi,\ n_\psi n_\phi) : A \to C$$

$\mathbb{F}_\Phi \to \mathcal{D}$ functor

$A \mapsto A$

$\phi \mapsto \phi_{\mathcal{D}}$

$\mathcal{D}$: base cat. of $\mathbb{F}_\Phi$



$\mathcal{C}$: cat.

covariant functor $\mathcal{C} \longrightarrow \mathbb{F}_{\Phi}$

(under some conditions on $\Phi, \mathcal{D}, \mathcal{C}$)

pre-Frobenioid structure on $\mathcal{C}$

We call $\mathcal{C}$ w/ pre-Frd str. pre-Frobenioid

Φ : division monoid

$\mathcal{D}$: base cat.

---

model Frobenioid

Def 7.23 (cf. [Frd I, Th 5.2])

$\Phi : \mathcal{D} \longrightarrow (\text{comm. Monoids})$ (divisorial) monoid on $\mathcal{D}$

$\mathbb{B} : \mathcal{D} \longrightarrow (\text{comm. Monoids})$ gp-like monoid on $\mathcal{D}$ $(\pm)$

$\forall A \in \mathrm{Ob}(\mathcal{D})$ $\mathbb{B}(A)^{\text{inv.}} = \mathbb{B}(A)$

$\mathrm{Div}_{\mathbb{B}} : \mathbb{B} \longrightarrow \Phi^{gp}$ hom.

(nat. tr.)

Image $=: \Phi^{\text{birat}} \subset \Phi^{gp}$

We def'n a cat. $\mathcal{C}$ as follows:

<u>Obj</u>: $A = (A_{\mathcal{D}}, \alpha)$ $\quad A_{\mathcal{D}} \in \mathrm{Ob}(\mathcal{D})$, $\alpha \in \Phi(A_{\mathcal{D}})^{gp}$

$$A_{\mathcal{D}} = \mathrm{Base}(A) \qquad \Phi(A) := \Phi(A_{\mathcal{D}})$$
$$\mathbb{B}(A) := \mathbb{B}(A_{\mathcal{D}})$$

<u>Morph</u> $\phi : A = (A_{\mathcal{D}}, \alpha) \longrightarrow B = (B_{\mathcal{D}}, \beta)$

$$\phi = \big(\underset{\mathbb{N}_{\geq 1}}{\underset{\cap}{\deg_{Fr}(\phi)}},\ \underset{\mathrm{Morph}(\mathcal{D})}{\underset{\cap}{\mathrm{Base}(\phi)}} : A_{\mathcal{D}} \to B_{\mathcal{D}},\ \underset{\Phi(A)}{\underset{\cap}{\mathrm{Div}(\phi)}},\ \underset{\mathbb{B}(A)}{\underset{\cap}{u_\phi}} \big)$$

s.t. $\deg_{Fr}(\phi)\alpha + \mathrm{Div}(\phi) = \Phi(\mathrm{Base}(\phi))(\beta) + \mathrm{Div}_{\mathbb{B}}(u_\phi)$

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<u>composition</u>

$$\psi \circ \phi = \big(\deg_{Fr}(\psi)\deg_{Fr}(\phi),\ \mathrm{Base}(\psi) \circ \mathrm{Base}(\phi),$$
$$\Phi(\mathrm{Base}(\phi))(\mathrm{Div}(\psi)) + \deg_{Fr}(\psi)\,\mathrm{Div}(\phi),$$
$$\mathbb{B}(\mathrm{Base}(\phi))(u_\psi) + \deg_{Fr}(\psi)\, u_\phi \big)$$

$\mathbb{B} \ni f \mapsto \mathrm{div}(f)$

$\mathcal{C}$ is called the <u>model Frobenioid</u> def'd by the divisor monoid Φ, rat. fct. monoid $\mathbb{B}$

cf. [FrdI, §6]

$\mathbb{B}(X)^{\circ}$

$\mathbb{B}(X)$

§7.5.2 Frobenioid assoc. to Theta Function

Def 7.24 ([EtTh, Def 3.3])

$$\mathcal{D}_{\underline{X}} := \mathcal{B}^{\mathrm{temp}}(\underline{X})^0 \leftarrow \text{conn. obj.}$$

left adj. $\left(\mathcal{D}_{\underline{X}} \supset \mathcal{D}^{\mathrm{ell}}_{\underline{X}} \right)$ $\leftarrow$ unram. over cusps

$\mathrm{Hom}_{\pi_1(\mathcal{D})\text{-sets}}(B, A)$

$\simeq \mathrm{Hom}_{\pi_1(\mathcal{D}^{\mathrm{ell}})\text{-sets}}(B \otimes_{\pi_1(\mathcal{D})} \pi_1(\mathcal{D}^{\mathrm{ell}}), A)$

$\mathrm{Ob}(\mathcal{D}_{\underline{X}}) \ni Y$

$$\Phi_0(Y) := \varinjlim_{Z_{i,\infty} \to Y} \mathrm{Div}_+(\mathfrak{Z}_{i,\infty})^{\mathrm{Gal}(Z_{i,\infty}/Y)}$$

eff. Cartier div w/ support $\subset$ special fiber & cusps

$Y \leftarrow \mathfrak{Z}_i \leftarrow \mathfrak{Z}_{i,\infty}$

$Y \leftarrow Z_i \leftarrow Z_{i,\infty}$

$\Delta_Y \supset \{(\Delta_Y)_i\}_{i \in I}$ s.t.

filtr. char. open subgr. by

- $\bigcap_{i \in I} (\Delta_Y)_i = \{1\}$
- $\forall (\Delta_Y)_i \underset{\text{char. free \& min.}}{\overset{\exists}{\supset}} (\Delta_Y)_i^{\infty}$
- $\Delta \underset{\text{open}}{\supset} {}^{\forall} H$, $\exists i_H \in I$ s.t. $(\Delta_Y)_{i_H}^{\infty} \subset H$ & $i \in I$ $(\Delta_Y)_i^{\infty} \subset H \Rightarrow (\Delta_Y)_i^{\infty} \subset (\Delta_Y)_{i_H}^{\infty}$

$\mathcal{B} \ni f$

$\dim(f)$

$\pi_1(\mathrm{Fund}\Sigma, \xi_?)$

$\mathcal{B}(X)^0$

$\mathcal{B}(X)$

$$\mathbb{B}_\circ(Y) := \varinjlim_{Z_{Y,\infty}} \mathrm{Mero}(\mathfrak{Z}_{Y,\infty})^{\mathrm{Gal}(Z_{Y,\infty}/Y)}$$

$\cup$ $\qquad\uparrow$

$\mathbb{F}_\circ$ const. $\qquad$ (non-zero log-mero fct f s.t. for $\forall n \geq 1$, f has a n-th root in a temp. cover.)

$$\begin{array}{ccc} \mathbb{F}_\circ & \longmapsto & \mathrm{Im} =: \Phi_\circ^{\mathrm{cnst}} \\ \cap & & \cap \\ \mathbb{B}_\circ & \longmapsto & \Phi_\circ^{\mathrm{gp}} \\ & & \cup \\ & \longmapsto & \mathrm{Im} =: \Phi_\circ^{\mathrm{birat}} \end{array}$$

$\Phi_{\underline{X}} := \Phi_\circ^{\mathrm{perf}}$ perfection

$$\Phi_{\underline{X}}\Big|_{\mathcal{D}_{\underline{X}}^{\mathrm{ell}}} \supset \Phi^{\mathrm{ell}}_{\underline{X}^{\mathrm{ell}}} : A \longmapsto \left(\text{perf-saturation of } \left(\varinjlim_{Z_{Y,\infty} \to A} \mathrm{Div}_+(\mathfrak{Z}_{Y,\infty})\right)^{\mathrm{Gal}(Z_{Y,\infty}/A)} \subset \Phi_{\underline{X}}(A) \right)$$

in $\Phi_{\underline{X}}(A)$

i.e. $(\text{above})^{\mathrm{perf}} \cap \Phi_{\underline{X}}(A) =: \Phi^{\mathrm{ell}}_{\underline{X}^{\mathrm{ell}}}(A)$

$$\Phi^{\mathrm{ell}}_{\underline{X}} := \Phi^{\mathrm{ell}}_{\underline{X}^{\mathrm{ell}}}\Big|^{\mathcal{D}_{\underline{X}}} \subset \left(\Phi_{\underline{X}}\Big|_{\mathcal{D}_{\underline{X}}^{\mathrm{ell}}}\right)\Big|^{\mathcal{D}_{\underline{X}}} \subset \Phi_{\underline{X}}$$

left adj of $\mathcal{D}^{\mathrm{ell}}_{\underline{X}} \subset \mathcal{D}_{\underline{X}}$

Def. 7.25 ([E+Th, Def. 3.6])

$$\underline{\Phi}^{\mathrm{bs\text{-}fld}} := (\mathbb{R}\underline{\Phi}_0^{\mathrm{cnst}})|_{\mathcal{D}_{\underline{\underline{X}}}} \underset{(\underline{\Phi}_0^{\mathbb{R}})^{\mathrm{op}}}{\times} \underline{\Phi}_{\underline{\underline{X}}}^{\mathrm{ell}} \subset \underline{\Phi}_0^{\mathbb{R}}|_{\mathcal{D}}$$

$\mathbb{R}'\mathrm{d}$

$(\mathbb{Z}_{\geq 0} \rightsquigarrow \mathbb{R}_{\geq 0})$

$$\mathbb{F} := \mathbb{F}_0|_{\mathcal{D}} \underset{\underline{\Phi}_0^{\mathbb{R}}|_{\mathcal{D}}}{\times} (\underline{\Phi}_{\underline{\underline{X}}}^{\mathrm{ell}})^{\mathrm{op}} \to (\underline{\Phi}_{\underline{\underline{X}}}^{\mathrm{bs\text{-}fld}})^{\mathrm{op}}$$

$$\mathbb{B} := \mathbb{B}_0|_{\mathcal{D}} \underset{(\underline{\Phi}_0^{\mathbb{R}}|_{\mathcal{D}})^{\mathrm{op}}}{\times} (\underline{\Phi}_{\underline{\underline{X}}}^{\mathrm{ell}})^{\mathrm{op}} \to (\underline{\Phi}_{\underline{\underline{X}}}^{\mathrm{ell}})^{\mathrm{op}}$$

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$$(\mathcal{D}_{\underline{\underline{X}}}, \underline{\Phi}_{\underline{\underline{X}}}^{\mathrm{ell}}, \mathbb{B}, \mathbb{B} \to (\underline{\Phi}_{\underline{\underline{X}}}^{\mathrm{ell}})^{\mathrm{op}})$$

$\rightsquigarrow$ model Fr'd $\boxed{\underline{\mathcal{F}}}$

$$(\mathcal{D}_{\underline{\underline{X}}}, \underline{\Phi}^{\mathrm{bs\text{-}fld}}, \mathbb{F}, \mathbb{F} \to (\underline{\Phi}^{\mathrm{bs\text{-}fld}})^{\mathrm{op}})$$

$\rightsquigarrow$ model Fr'd $\underline{\mathcal{F}}^{\mathrm{bs\text{-}fld}}$

We call <u>base-field theoretic hull</u> of $\underline{\mathcal{F}}$

\mathcal $\mathcal{F}$
\mathfrak $\mathfrak{F}$

$\mathfrak{s}_{lN} \in \Gamma(\mathfrak{Z}_{lN}, \mathcal{L}_{lN}|_{\mathfrak{Z}}) \xrightarrow{\sim} \mathcal{O}_{\ddot{\mathfrak{Z}}_{lN}} \to \ddot{\mathcal{L}}_{lN}|_{\ddot{\mathfrak{Z}}_{lN}}$

$\tau_{lN} \in \Gamma(\ddot{\mathfrak{Y}}_{lN}, \ddot{\mathcal{L}}_{lN}) \xrightarrow{\sim}$

$\mathfrak{s}^{\sqcup}_N$, $\mathfrak{s}^{\sqcap}_N$ morph in $\underline{\mathcal{F}}$

$A_{\circledcirc} := \mathcal{O}_{\ddot{\underline{\mathfrak{Y}}}}$

$\Big) ^{\mathrm{gp}} \to \left(\overline{\mathfrak{F}}^{\mathrm{ell}}_{\underline{\underline{X}}} \right)^{\mathrm{gp}}$

$\underline{\mathrm{Prop}\ 7.26}$ (cf. [EtTh, Prop 4.2, Prop 4.3, Prop 5.2])

"cut this"

(1). $(\mathfrak{s}^{\sqcup}_N, \mathfrak{s}^{\sqcap}_N)$ is a "lN-th root of a right fraction pair of $(\circledcirc)$"

$$\begin{tikzcd} A_N \arrow[r, "\mathfrak{s}^{\sqcap}_N"] \arrow[d, "\alpha"] \arrow[dr, phantom, "\curvearrowright"] & B_N \arrow[d, "\beta"] \\ A_{\circledcirc} \arrow[r, "\mathfrak{s}^{\sqcap}_1"] & B_{\circledcirc} \end{tikzcd} \qquad \begin{tikzcd} A_N \arrow[r, "\mathfrak{s}^{\sqcup}_N"] \arrow[d, "\alpha"] \arrow[dr, phantom, "\wr"] & B_N \arrow[d, "\beta"] \\ A_{\circledcirc} \arrow[r, "\mathfrak{s}^{\sqcup}_1"] & B_{\circledcirc} \end{tikzcd}$$

$B_{\circledcirc}$ from $\ddot{\mathcal{L}}_l|_{\ddot{\mathfrak{Z}}_l}$

α, β Frob. deg $= N$

(. . . . . .)

$\frac{\mathrm{hull}}{\mathcal{F}}$

(2),

(3),

(2). $\Pi^{\text{tp}}_{X} \curvearrowright \mathcal{L}_{lN} \otimes_{\mathbb{Z}_{lN}} \mathcal{O}_{\mathfrak{J}_{lN}}$, $\Pi^{\text{tp}}_{Y} \curvearrowright \mathcal{L}_{lN}$

$\quad\; \rotatebox{90}{$\subseteq$} \qquad\qquad\qquad\qquad\qquad\quad \rotatebox{90}{$\supseteq$}$

$\quad S_{lN} \qquad\qquad\qquad\qquad\qquad\quad T_{lN}$ actions coincide w/ "the bi-Kummer lN-th root of Θ" arising from (1). "N-th root of $\Theta^{1/l}$"

cat this

(3). the Kummer class det'd by the "bi-Kummer lN-th root" in (2)

is equal to $\ddot{\eta}^{\Theta}$ mod lN rel. to nat. isom $\Delta_\Theta \otimes \mathbb{Z}/lN \cong \mu_{lN}(-)$

" " " N-th root" in (2)

is equal to $\ddot{\eta}^{\Theta}$ mod N " " $l\Delta_\Theta \otimes \mathbb{Z}/N \cong \mu_N$

Def 7.28 ([EtTh, Def 5.4])

$S \in \mathrm{Ob}(\mathfrak{T})$ (l, N)-theta saturated

$\overset{\text{def}}{\Leftrightarrow}$ $\begin{cases} S : \mu_{lN}\text{-saturated i.e. } \mu_{lN}(S) \cong \mathbb{Z}/lN \text{ as abst. gp} \\ \#(l\Delta_\Theta)_S \otimes \mathbb{Z}/N = N \end{cases}$

$\mu_{lN}(S) \subset \mathrm{Aut}_{\mathfrak{T}}(S)$

$A \in \mathrm{Ob}(\mathcal{D}_{\underline{X}})$ $\quad \Pi^{\text{tp}}_{\underline{X}} \twoheadrightarrow (\Pi^{\text{tp}}_{\underline{X}})^{\Theta}$ $\quad \Delta_\Theta \subset (\Pi^{\text{tp}}_{X})^{\Theta}$ $\rightsquigarrow$ $(l\Delta_\Theta)_A \subset \mathrm{Aut}^{\Theta}_{\mathcal{D}}(A)$

$\mathrm{Aut}_{\mathcal{D}}(A) \twoheadrightarrow \mathrm{Aut}^{\Theta}_{\mathcal{D}}(A)$,

$A \in \mathrm{Ob}(\mathfrak{T})$ $\quad (l\Delta_\Theta)_A := (l\Delta_\Theta)_{A^{\text{bs}}}$

$A \mapsto A^{\text{bs}} \in \mathrm{Ob}(\mathcal{D}_{\underline{X}})$

Prop 7.29 ([E+Th, Prop 5.5])

the Kummer class in Prop 7.26 det's

$\forall\, \mathcal{S} \in \mathrm{Ob}(\mathcal{I})$ (l, N)-theta sat.

functorial $(l\Delta_\Theta)_{\mathcal{S}} \otimes \mathbb{Z}/N \xrightarrow{\sim} \mu_N(\mathcal{S})$

w.r.t. linear morph of (l, N)-theta sat. obj's

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Th 7.30 ([E+Th 5.6], cat. theoreticity of Fr'd theoretic ayl. rig.)

$\Phi : \mathcal{I} \longrightarrow \mathcal{I}$ self cat. equiv.

$\Rightarrow$ Φ : preserves (l, N)-theta sat. obj.

for such obj's $\mathcal{S}$, Φ preserves

$(l\Delta_\Theta)_{\mathcal{S}} \otimes \mathbb{Z}/N \xrightarrow{\sim} \mu_N(\mathcal{S})$

Th 7.31 ([E+Th, Th 5.7], cat. theoreticity of Fr'd theoretic theta fct)

$\Phi : \underline{\underline{X}} \xrightarrow{\sim} \underline{\underline{X}}$ self cat. equi.

Φ preserves "right pairs of l-th root of $\ddot{\Theta}(\sqrt{-1})^{-1}\,\ddot{\Theta}$"

up to μ_{2l} & transl. by $l\underline{\mathbb{Z}} = \mathrm{Gal}(\ddot{\underline{Y}}/\ddot{X})$)

Th 7.34 ([E+Th 5.10] (cat. theoreticity of Fr'd theoretic theta env.))

$\underline{\underline{X}} \rightsquigarrow A$,

$s_N^{\Pi}, s_N^{\sqcup} : A_N \longrightarrow B_N$

$\rightsquigarrow s_{\Theta N}, \tau_{\Theta N}$ ("N-th root of $\ddot{\Theta}^{1/l}$")

$$\mathbb{E}_N := s_N^{\Pi\text{-gp}}\left(\mathrm{Im}\{\Pi^{\mathrm{tp}}_{\underline{\underline{Y}}} \subset \Pi^{\mathrm{tp}}_{\underline{\underline{X}}} \twoheadrightarrow \mathrm{Aut}_{\underline{\underline{X}}}(B_N^{\mathrm{bs}})\}\right)\mu_N(B_N) \subset \mathrm{Aut}_{\underline{\underline{X}}}(B_N)$$

$$\mathbb{E}_N^{\Pi} := \mathbb{E}_N \times_{\mathrm{Im}(\Pi^{\mathrm{tp}}_{\underline{\underline{Y}}})} \Pi^{\mathrm{tp}}_{\underline{\underline{Y}}}$$

$\rightsquigarrow \xi : \mathbb{E}_N^{\Pi} \longrightarrow \mathrm{Aut}_{\underline{\underline{X}}}(B_N)$ μ_N-outer hom.

$\rightsquigarrow$ nat. proj. $\mathbb{E}_N^{\Pi} \longrightarrow \mathbb{E}_N$

($\rightsquigarrow$ hom up to inner aut. dut'd by an elt $\in \ker(\mathbb{E}_N^{\Pi} \twoheadrightarrow \Pi^{\mathrm{tp}}_{\underline{\underline{X}}})$ or $\mu_N(B_N) \subset \mathrm{Aut}_{\underline{\underline{X}}}(B_N)$)

(1). $\Phi : \underline{\Phi} \xrightarrow{\sim} \underline{\Phi}$ self cat.-equi.

preserves isom classes of A_N, B_N ...

(2) take $\beta : \Phi(B_N) \xrightarrow{\sim} B_N$ isom.

$\rightsquigarrow \Phi_{Aut} : Aut_{\underline{\Phi}}(B_N) \xrightarrow{\sim} Aut_{\underline{\Phi}}(B_N)$,

induce ($\Phi^{birat}_{Aut} : Aut_{\underline{\Phi}^{birat}}(B_N^{birat}) \xrightarrow{\sim} Aut_{\underline{\Phi}^{birat}}(B_N^{birat})$)

$\therefore$ preserve $\mathcal{O}^\times(B_N)$, $(\mathcal{O}_K^\times)^{1/N} \subset \mathcal{O}^\times(B_N)$, $\mathrm{Im}(s_N^{\Pi\text{-}gp})$, $\mathrm{Im}(s_N^{\sqcup\text{-}gp})$

map the data $\mathbb{E}_N$ ($\subset Aut_{\underline{\Phi}}(B_N)$),

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bi-Kummer N-th root

$$A_N \xrightarrow{s_N^{\Pi}} B_N \qquad s_N^{\Pi\text{-}gp} : H_{B_N} \to Aut_{\underline{\Phi}}(B_N)$$

$$s_N^{hs} \downarrow \qquad \downarrow \beta$$

$$A_\Theta \xrightarrow{s_N^{\Pi}} B_\Theta$$

A_Θ^{hs} $\quad \Pi_X^{tp} \to Aut_{\mathcal{D}}(A^{hs})$

$\Pi_X^{tp} \twoheadrightarrow H_\Theta \quad H_\Theta \to \underbrace{H_{A^{hs}}}_{def}$

$G_K \twoheadrightarrow H_\Theta^{ss\text{-}fld}$

$H_A \subset Aut_{\underline{\Phi}}(A)/\mathcal{O}^\times(A)$

$\downarrow\wr \qquad \cap$

$H_{A^{hs}} \subset Aut_{\mathcal{D}}(A^{hs})$

to data $\delta_1 \mathbb{E}_N \delta_1^{-1}$ ($\subset Aut_{\underline{\Phi}}(B_N)$)

$\delta_1 \mathrm{Im}(s_N^{\Pi\text{-}gp}) \delta_1^{-1}$

$\delta_1 \delta_2 \delta_3 \mathrm{Im}(s_N^{\sqcup\text{-}gp}) \delta_3^{-1} \delta_2^{-1} \delta_1^{-1}$

for some $\delta_1 \in \mathcal{O}^\times(B_N)$

$\delta_2 \in \mu_{2N}(B_N) \cap (\mathcal{O}_K^\times)^{1/N}$ ($\subset \mathcal{O}^\times(B_N)$)

$\delta_3 \in s_N^{\Pi\text{-}gp}(Aut_{\mathcal{D}}(B_N^{hs}))$

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fct)

"

$= Gal(\ddot{Y}/\ddot{X})$)

m.

B_N)

$Aut_{\underline{\Phi}}(B_N)$)

... mines out the ... by an elt ... ($\mathbb{E}_A^{\pi} \to \Pi_C^{tp}$ ... $(B_N) < Aut_{\underline{\Phi}}(B_N)$)

next week

1F
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(3)

[IUTch II, Prop 1.2]

conj$(\beta)\circ\Psi$ preserves the $\mathrm{Aut}_{\Psi}(B_N)$-orbit of

$$\xi : \mathbb{E}^{\Pi}_N \longrightarrow \mathrm{Aut}_{\Psi}(B_N)$$

$\left(\begin{array}{c} s^{\Pi\text{-gp}}_N, \ s^{\ell\text{-gp}}_N \\ s^{\Pi\text{-}\Pi}_N \quad s^{\ell\text{-}\Pi} \\ \Pi^{\text{tp}}_{\underline{Y}} \longrightarrow \mathbb{E}^{\Pi}_N \end{array} \right)$

in a fashion which is compat w/

the mono-theta env. str. on $\mathbb{E}^{\Pi}_N$ involving $s^{\ell\text{-}\Pi}_N$

More precisely

$$\begin{array}{ccc} \mathbb{E}^{\Pi}_N & \xrightarrow[\sim]{\gamma} & \ddot{\mathbb{E}}^{\Pi}_N \\ \downarrow \xi & \circlearrowleft & \downarrow \kappa\cdot\xi \\ \mathrm{Aut}_{\Psi}(B_N) & \xrightarrow[\sim]{\Psi_{\mathrm{Aut}}} & \mathrm{Aut}_{\Psi}(B_N) \end{array}$$

κ: inner autom. of $\mathrm{Aut}_{\Psi}(B_N)$

$\left(\mathbb{E}^{\Pi}_N \xrightarrow{\sim} \Pi^{\text{tp}}_{\underline{Y}}[\mu_N] \right)$

γ: autom. of tp. gp which det's an autom. of mono-theta env. & compat w/ the $\Pi^{\text{tp}}_{\underline{X}}$-conj. class of Ψ^{bs} induced autom. of $\Pi^{\text{tp}}_{\underline{X}}$